Explosive growth of inhomogeneities in the distribution of droplets in a turbulent air

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Abstract

We study how the spatial distribution of inertial particles evolves with time in a random flow. We describe an explosive appearance of caustics and show how they influence an exponential growth of clusters due to smooth parts of the flow, leading in particular to an exponential growth of the average distance between particles.

Random compressible flows generally have regions where contractions accumulate and density grows. Infinitesimal elements expand or contract exponentially which can be characterized by the set of Lyapunov exponents. Since the sum of the exponents is non-positive [1, 2, 3, 4], density tends to a singular multi-fractal set with moments growing exponentially. Both the evolution and the final state of density in spatially smooth random flows have been described recently within some models [3, 4, 5, 6]. The flow of inertial particles is compressible even when the flow of ambient fluid is incompressible [7] so particles participate in the fractalization and have some of their concentration moments growing exponentially [4]. On the other hand, every time there is a negative velocity gradient exceeding the inverse viscous response time of particles, faster particles from behind catch slower ones creating folds in distribution and caustics [8, 9]. Such breakdowns of distribution lead to finite-time singularities and explosive growth of some density moments. The goal of the present paper is to describe the statistical evolution of concentration from a uniform one to a set of clusters and voids and, in particular, to describe the role of of folds in this evolution.

Because of folds, the problem of inertial particles in a flow is kinetic rather than hydrodynamic [8, 10]. Analytic approach to a realistic kinetic description does not seem to be feasible now. On the other hand, the significant progress of analytic Lagrangian methods [3] makes it tempting to use them: to follow, for instance, a couple of close particles and to account only for a local velocity gradient. The question is: what can we learn from the Lagrangian approach about the statistics of particle concentration? To answer that, one needs a model that allows to compare numerical data from kinetics with an analytic Lagrangian solution. For that end we consider here the motion of inertial particles in a one-dimensional random flow, which is appropriate for our main goal to describe the role of breakdowns that are one-dimensional in any space dimensionality. This model is a subject of much interest from different perspectives [11]. Here we briefly review what is known and derive new results, in particular, describe the statistics of the inter-particle distances R. We also carry direct numerical simulation of kinetics in this model and find the growth rates of the moments of concentration n. It is only for smooth flows that one can immediately convert R into n (in 1d simply taking $n = 1/|R|$). Since the flow of inertial particles has discontinuities, any given interval between two chosen particles does not contain the same particles all the time. Particles can enter and leave the interval i.e. numerous folds appear in particle distributions making nonlocal even the problem of describing single-point density statistics. We show

that indeed the growth rates of density moments and inter-particle distances are different.

Particle coordinate $\mathbf{q}$ and velocity $\mathbf{V}$ change according to $d\mathbf{q}/dt = \mathbf{V}(\mathbf{q},t)$ and $d\mathbf{V}/dt = [\mathbf{u}(\mathbf{q},t) - \mathbf{V}]/\tau$ with $\mathbf{q}(\mathbf{r},0) = \mathbf{r}$. Here the viscous (response) time is $\tau = (2/9)(\rho_0/\rho)(a^2/\nu)$ with a particle radius and ρ_0, ρ particle and fluid densities respectively. We treat the fluid velocity $\mathbf{u}$ as a given random function of time and smooth function of space coordinates. Let us briefly remind some relevant properties of smooth compressible random flows [3]. The behavior of an infinitesimal volume is governed by the local matrix of derivatives (called strain matrix) taken in the Lagrangian frame $s_{ik} = \partial u_i/\partial x_k$. Considering the distance between two fluid particles, $\mathbf{R}(t, \mathbf{r}_1 - \mathbf{r}_2) = \mathbf{q}(\mathbf{r}_1, t) - \mathbf{q}(\mathbf{r}_2, t)$ one finds $\langle R^m \rangle \sim \exp(E_m t)$ with E_m being a convex function of m. Density can be expressed as $n(t) = \det^{-1} \partial R_i(t, \mathbf{r})/\partial r_j$ (provided that the initial distribution is uniform $n_0 = 1$) so that the Lagrangian moments $\langle n^{-m} \rangle$ are related to space-averaged (Eulerian) moments via $\langle n^{-m} \rangle = \langle n^{1-m} \rangle_E \sim \exp(\Gamma_m t)$ (every trajectory comes with the weight n^{-1}). Therefore, $\Gamma_0 = 0 = \Gamma_1$ which correspond to conservation of mass and volume (Lagrangian and Eulerian measures) respectively. In one-dimensional (1d) smooth flows, $\Gamma_m = E_m$.

In 1d, one has for the distance $R(t)$ and velocity difference $v(t)$ between two close inertial particles:

$$\dot{R} = v\,, \quad \tau\dot{v} = sR - v \qquad \Rightarrow \qquad \tau\ddot{R} + \dot{R} = sR\,. \tag{1}$$

The substitution $R = \Psi \exp(-t/2\tau)$ turns (1) into Schrödinger equation with a random potential (Anderson localization), with space and the localization length replacing time and the Lyapunov exponent.

The quantity $\sigma = v/R$ satisfies the Langevin equation driven by the random noise $s(t)$

$$\dot{\sigma} = -\sigma^2 - \sigma/\tau + s/\tau \equiv -dU/d\sigma + s/\tau\,. \tag{2}$$

Let us describe the probability of finite-time singularity (explosion) $\sigma \to -\infty$ which corresponds to crossing of particle trajectories. Such probability can be written as a path integral over trajectories with $\sigma(0) = \sigma_0, \sigma(T) = -\infty$:

$$P(T) = \int \mathcal{D}\sigma\mathcal{D}p\mathcal{D}s\mathcal{P}\{s\} \exp\left\{ \int_0^T ip\left[\dot{\sigma} + W - \frac{s}{\tau}\right] dt' \right\}. \tag{3}$$

Here $\mathcal{P}\{s\}$ is the probability functional for s and $W = U' = (\sigma^2 + \sigma/\tau)$. When T is less than the average time between explosions (defined below), $P(T)$ is given by the single trajectory

(optimal fluctuation [12, 13, 14]) which maximizes the probability and can be found by a saddle-point integration of (8).

First, consider T which is much less than the correlation time of the air gradient s. Then the optimal fluctuation corresponds to $s = s_0$ which does not change during t. In this case the integration over the processes $s(t)$ is reduced to the averaging over a single value s_0 with the measure $P_s(s_0)$, which is a single-time statistic of velocity gradient s. The saddle-point integration over the fields p, σ is reduced to solving the equation (2) with constant $s(t) = s_0$ and the boundary conditions $\sigma(0) = \sigma_0, \sigma(T) = -\infty$. Straightforward integration yields the following relation:

$$\begin{aligned} T &= \tau \int_{\sigma_0}^{-\infty} \frac{d\sigma}{s_0 - \sigma - \tau\sigma^2} \\ &= \tau(-1 - 4s_0\tau)^{-1/2} \left[\pi - 2\arctan\left(\frac{1 + 2\sigma_0\tau}{\sqrt{-1 - 4s_0\tau}}\right)\right] , \end{aligned} \tag{4}$$

which formally gives a relation between the optimal value of s_0 and the collapse-time T. It is not possible to find the analytic expression for $s_0(T)$ for a general value of σ_0, however the situation greatly simplifies for $\sigma_0 = +\infty$. In this case the PDF $P(T)$ can be interpreted as the distribution of time intervals between consequent collapses. Note, that as long as the trajectory starting from $\sigma_0 = +\infty$ passes through all values of σ this distribution is also a lower estimate for the $P(T)$ for general σ_0. Substituting $\sigma_0 = +\infty$ in (4) one obtains

$$T = \frac{2\,\pi\tau}{\sqrt{-1 - 4s_0\tau}} \tag{5}$$

or equivalently

$$s_0 = -\frac{1}{4\tau} - \frac{\pi^2\tau}{T^2} \tag{6}$$

In this case the probability of collapse is given by

$$P(T) = P_s(s_0)\left|\frac{ds_0}{dT}\right| = \frac{2\pi^2\tau}{T^3} P_s\left(-\frac{1}{4\tau} - \frac{\pi^2\tau}{T^2}\right) . \tag{7}$$

One can see from this expression that collapses occur only if there is a finite probability of having sufficiently negative flow gradient, $s < -1/4\tau$. In particular for Gaussian gradients, $P_s(x) = (\alpha/\pi)^{1/2}\exp(-\alpha x^2)$, the short-time asymptotics is as follows: $P(T) \sim T^{-3}\exp(-\alpha\pi^2\tau^2/T^4)$.

Consider now the case when the correlation time of s is much shorter than T. In this case, the noise can be effectively considered as white Gaussian, $\langle s(t)s(0)\rangle = 2D\tau^2\delta(t)$, and

$$P(T) = \int \mathcal{D}\sigma \exp\left\{-\frac{1}{4D}\int_0^T [\dot\sigma + W]^2 dt'\right\} . \tag{8}$$

For $DT^3 \ll 1$, if follows from the saddle point approximation that the probability is given by the optimal fluctuation (also called "instanton" trajectory [12, 13, 14]) which satisfies $\ddot{\sigma} = W(\sigma)W'(\sigma)$ with the boundary conditions $\sigma(0) = \sigma_0, \sigma(T) = -\infty$. After one integration one obtains the following equation:

$$\dot{\sigma} = -\sqrt{E + W^2} \tag{9}$$

where E is an integration constant, characterizing the trajectory. This constant is determined by the boundary conditions:

$$T = \int_{-\infty}^{\sigma_0} \frac{d\sigma}{\sqrt{E + W^2}} \tag{10}$$

The probability of such fluctuation is given by $P(T) \sim \exp(-A)$, where

$$A = \int_0^T dt \frac{(\dot{\sigma} + W)^2}{4D} = \int_{-\infty}^{\sigma_0} \frac{d\sigma}{4D} \frac{(\sqrt{E + W^2} - W)^2}{\sqrt{E + W^2}} \tag{11}$$

Unfortunately, the integrals (10,11) can not be expressed through known special functions, so we are able to get analytical results only in some limiting cases. We will consider the case $\sigma_0 = +\infty$ following the same arguments as in the preceding analysis. First, we consider the limit $E\tau^4 \ll 1$ which as follows from (10) corresponds to large times $T \sim \tau \log(1/E\tau^4) \gg \tau$. From the expression (11) we have in the main order:

$$A = \int_{-\infty}^{\infty} \frac{(|W| - W)^2 d\sigma}{4D|W|} = \int_{-1/\tau}^{0} \frac{|W| d\sigma}{D} = \frac{1}{6D\tau^3}\,. \tag{12}$$

We see, that in the main approximation the action does not depend on the T, which has a simple interpretation: the collapses are produced by universal tunneling processes, each having a probability $\exp(-1/6D\tau^3)$ and characteristic time-scale τ. In order to find the T dependence of the total probability we should study the fluctuations around this instanton [16] which would involve some bulky calculations. However, for the intermediate region of $T \ll \tau \exp(1/6D\tau^3)$ one can treat these tunnelings as a Poissonian process and predict the linear behavior $P(T) \sim T/\tau \exp(-1/6D\tau^3)$. This expression is certainly not true in the case $D\tau^3 \lesssim 1$ when the action A is not large and the saddle-point approximation is not applicable. Another limiting case, which can be studied analytically corresponds to the very high "energies" $E\tau \gg 1$ where one can neglect the linear σ/τ terms in (10,11), so that one has

$$T = \frac{\Gamma(1/4)^2}{2\sqrt{\pi}E^{1/4}}, \qquad A = \frac{\Gamma(1/4)^8}{96\pi^2 DT^3} \approx \frac{31.5}{DT^3} \tag{13}$$

The crossover between the two regimes happens at $T \sim \tau$. To summarize, for the white $s(t)$ one gets

$$P(T) \sim \begin{cases} \exp(-c/DT^3), & T < \tau, \\ T\exp(-1/6D\tau^3) & \tau < T < \tau\exp(1/6D\tau^3), \end{cases} \tag{14}$$

where $c = [\Gamma(1/4)]^8/96\pi^2 \approx 31.5$.

Since we consider dilute distribution of particles and neglect their pressure, then σ changes sign after the explosion as the fast particle overcomes the slow one. That is the flux of probability that goes to $\sigma \to -\infty$ returns from $\sigma \to +\infty$. That allows for the steady-state probability density function (PDF) having constant probability F flux equal to the number of breakdowns per unit time. Such PDF must have $P(\sigma) \approx F\sigma^{-2}$ at $\sigma \to \pm\infty$. If, as is usually the case, the initial $P(\sigma, 0)$ does not have power tails, they appear at $t = +0$ according to $P(t, \sigma) \propto P(t)\sigma^{-2}$ and (7,14).

When $\sigma \to -\infty$, $R \to 0$. To establish the sufficient condition for negative moments of the distances to blow-up in a finite time, introduce $R_{l,k} = \langle \sigma^l R^k \rangle$. Assuming even k, using (2) and Cauchy inequality $R_{1,k} \le R_{2,k}^{1/2} R_{0,k}^{1/2}$ we get for $Z = R_{0,k}^{1/k}$ the majoring inequality $k(Z_{tt} + Z_t/\tau) \ge 0$. For positive k, it means smooth evolution with Z growing. For negative k, this inequality gives $Z(t) \le Z(t_1) + \tau Z_t(t_1)(1 - e^{-(t-t_1)/\tau})$. This means that Z turns into zero, and respectively, the negative momenta of the distances ($k < 0$) will blow up in a finite time if at some t_1: $Z + \tau Z_t < 0$ or in other terms, $\tau dR_{0,k}/dt > |k|R_{0,k}$. This condition is readily satisfied for most random processes $s(t)$, the detailed analysis will be published elsewhere.

In the rest of the paper we approximate the flow gradient $s(t)$ in the particle reference frame by a white noise, which is quantitatively good for heavy particles and give a qualitatively correct description in other cases. In the white case, a variety of analytic results can be obtained, some translated from the localization theory and super-symmetric quantum mechanics [15, 16] and some original that we derive here. The steady-state PDF can be found explicitly [15]

$$P_0 = \frac{F}{D}\exp\left[-\frac{U(\sigma)}{D}\right]\int_{-\infty}^{\sigma}\exp\left[\frac{U(\sigma')}{D}\right]d\sigma' \,, \tag{15}$$

with the flux $F = D\partial P_0/\partial\sigma + (\sigma^2 + \sigma/\tau)\,P_0 \approx (2\pi\tau)^{-1}\exp[-1/(6D\tau^3)]$ for $D\tau^3 \ll 1$ (the dimensionless Stokes number $D\tau^3 = St$ measures the inertia of the particle). At $St \gg 1$,

$F \approx 0.2D^{1/3}$ [11], note that the average time between breakdowns is much smaller than τ in this limit. The Lyapunov exponent $\langle\sigma\rangle$ changes sign at $St_* \approx 0.827$ [11]: $\langle\sigma\rangle \approx -D\tau^2/2$ at $St \ll St_*$ and $\langle\sigma\rangle \sim D^{1/3}$ at $St \gg St_*$. That means that small particles cluster while large ones mix uniformly.

Note that the Gibbs state $\exp(-U/D)$ is non-normalizable in this case. The flux state (15) minimizes entropy production [17]. It can be shown that it is indeed the asymptotic solution at $t \to \infty$ [18].

To describe the joint statistics of σ and R we introduce the generating function $Z_k(\sigma, t) = \langle \delta\left[\sigma\left(t\right) - \sigma\right] R^k(t) \rangle$, which satisfies the equation

$$\frac{\partial Z_k}{\partial t} = k\sigma Z_k + \frac{\partial}{\partial\sigma}\left(\frac{\sigma}{\tau} + \sigma^2 + D\frac{\partial}{\partial\sigma}\right) Z_k\,. \tag{16}$$

Substitution $Z_k = \Psi(\sigma, t)\exp[-U/2D]$ turns it into the Schrödinger equation in a double well, which has been a subject of numerous works related to tunnelling and instantons (see e.g. [16, 19, 21, 22]. Following [16, 19] we first find (non-normalizable) solutions $\exp(\gamma_k t/\tau - U/D) f_k(\sigma)$ with f_k being polynomials and then the conjugated solutions by the method of variable constants. For example, there are steady states $Z_0 = P_0$ and

$$Z_1(\sigma) = (1 + \sigma\tau)\exp\left[\frac{U(\sigma)}{D}\right]\int_{-\infty}^{\sigma}\exp\left[\frac{U(\sigma')}{D}\right]\frac{d\sigma'}{(1+\sigma'\tau)^2}\,.$$

In particular, this solution allows one to obtain the mean velocity difference between two particles at the distance a: $a\int \sigma Z_1(\sigma, t)\, dt$ needed, for instance, to calculate the collision rate. The growth rates of the moments of inter-particle distance can be obtained from (16) or in a straightforward way by writing

$$\dot{R}_{l,k} = -lR_{l,k}/\tau - (l-k)R_{l+1,k} + l(l-1)DR_{l-2,k}\ , \tag{17}$$

where the higher moments are expressed only via lower ones. Assuming that for a given k all $R_{l,k} \propto \exp(\gamma_k t)$ we get for γ_k the $(k+1)$-st order algebraic equation. For the second moment one gets $\gamma_2\left(\gamma_2 + \tau^{-1}\right)\left(\gamma_2 + 2\tau^{-1}\right) - 4D = 0$ which gives $\gamma_2 \approx 2D\tau^2$ for $D\tau^3 \ll 1$ and $\gamma_2 \approx (4D)^{1/3}$ for $D\tau^3 \gg 1$.

For arbitrary k, we find asymptotics. If $D\tau^3k^2 \ll 1$ then $\gamma_k \approx D\tau^2 k(k-1)/2$. When $D\tau^3k^2 \gg 1$, the determinant of (17) is approximately $\gamma_k^{k+1} - \gamma_k^{k-2}Dk(k-1)\sum$, where $\sum = \sum_1^k i(k-i) \propto k^2$ and $\gamma_k \propto (Dk^4)^{1/3}$. Let us compare the growth rates of the distance moments for the inertial particles with those for smooth compressible short-correlated flow.

For the latter, $\gamma_k \sim k(k-1)$ while for the former the dependence is parabolic only for low-order moments in the low-inertia limit $D\tau^3 k^2 \ll 1$. High moments correspond to high inertia and have $\gamma_k \sim (Dk^4)^{1/3}$ even for $St \ll 1$. Note that conservation requires $\gamma_0 = \gamma_1 = 0$ for inertial particles as well.

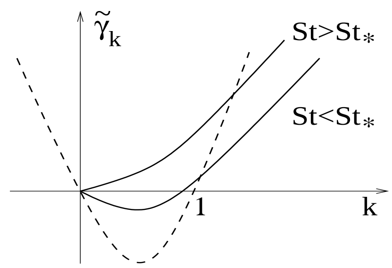

FIG. 1: Growth rates of distance moments for a smooth flow (broken line) and inertial particles for different Stokes numbers (two solid lines).

Since R is sign-changing for inertial particles, the statistics of $|R|$ deserves separate study, particularly for comparison with the concentration. The equation for the time derivative of $\tilde{R}_{lk} = \langle \sigma^l |R|^k \rangle$ differs from (17) by the extra term $2\langle \sigma^{l+1} R^{k+1} \delta(R) \rangle$, which is nonzero for $l = k$. As a result, the growth rates $\tilde{\gamma}_k \equiv \tilde{R}_{lk}^{-1} d\tilde{R}_{lk}/dt$ differ remarkably from γ_k. The most dramatic new effect can be readily appreciated since $\tilde{\gamma}_k$ are related to the Lyapunov exponent via $\langle \sigma \rangle = (d\tilde{\gamma}_k/dk)_{k=0}$. At high inertia, when $St > St_*$ and $\langle \sigma \rangle$ is positive, it is thus evident that $\tilde{\gamma}_1 > 0$ as seen from the sketch in Fig. 1. Nonzero growth rate of $\langle |R| \rangle$ is a remarkable qualitatively new effect with a clear physical meaning: in every breakdown, extra particles enter the interval between the two particles that we follow; the interval length must grow to ensure conservation of the total number of particles. From this interpretation, it is clear that the growth rate must be nonzero at low inertia as well, when it must be proportional to the exponentially small rate of explosions: $\gamma_1 \sim F \sim \exp(-1/6St)$. Remarkably, one can also establish asymptotically exact pre-exponential factor. Consider $d|R|/dt = \sigma|R| + 2\delta(R)\sigma R^2$. The growth of $\langle |R| \rangle$ must be determined by the last term, which accounts for the breakdown processes, since $\langle R \rangle$ does not grow. In order to obtain the explicit expression for $\tilde{\gamma}_1$ we first analyze the dynamical equation on the stages between the breakdowns, which formally coincides for both R and $|R|$ and then account for breakdowns explicitly. For delta-correlated s, we can break the time interval into pieces with independent evolution (markovian property). Using this fact and multiplicative nature of (1) one can

derive the following identity

$$\left\langle \frac{|R(t)|}{|R(0)|} \right\rangle = \left\langle \frac{|v(t_1)|}{|R(0)|} \right\rangle \left\langle \frac{|R(t)|}{|v(t_N)|} \right\rangle \prod_{k=2}^{N} \left\langle \frac{|v(t_k)|}{|v(t_{k-1})|} \right\rangle \tag{18}$$

Here $t_1..t_N$ are the times when the breakdowns happened, $v(t_k)$ are absolute values of the velocities in these breakdowns. All the averages in this expression correspond to the dynamics between the breakdowns, for which (1) can be solved explicitly. In order to study the dynamics between breakdowns we introduce the function $Z(\sigma, \sigma_0, t)$ which is the solution of (16) with $k = 1$ and the initial condition $Z(\sigma, \sigma_0, 0) = \delta(\sigma - \sigma_0)$. In contrast to the previous analysis we consider different boundary conditions for the function $Z(\sigma, \sigma_0, t)$. Namely we will assume that there is no flux at $\sigma = +\infty$, which means that $Z(\sigma, \sigma_0, t)$ decays exponentially there. In this case $Z(\sigma, \sigma_0, t)$ can be interpreted as $Z(\sigma, \sigma_0, t) = \overline{R(t)/R(0)}$ where the averaging is performed only on the trajectories which had no breakdown up to time t and which satisfy the following boundary conditions $\sigma(t) = \sigma, \sigma(0) = \sigma_0$. Note, that for such trajectories we have $|R(t)/R(0)| = R(t)/R(0)$. We are able to fix the breakdown moment at t by taking the limit $\sigma \to -\infty$. Analogously the limit $\sigma_0 \to +\infty$ fixes the preceding breakdown moment at $t = 0$. In order to analyze the product (18) we introduce three new functions:

$$\frac{|v(t_k)|}{|R(t_k - t)|} = J_+(\sigma_0, t) = -\sigma^3 Z(\sigma, \sigma_0, t)|_{\sigma \to -\infty} \tag{19}$$

$$\frac{|R(t_k + t)|}{|v(t_k)|} = J_-(\sigma, t) = \sigma_0^{-1} Z(\sigma, \sigma_0, t)|_{\sigma_0 \to \infty} \tag{20}$$

$$\frac{|v(t_{k+1})|}{|v(t_k)|} = M(t_{k+1} - t_k) = \sigma_0^{-1} J_+(\sigma_0, t_{k+1} - t_k)|_{\sigma_0 \to \infty} \tag{21}$$

These function have the following meaning. $J_+(\sigma_0, t)dt$ is an average ratio of $|v(t)/R(0)|$ for trajectories with the boundary condition $\sigma(0) = \sigma_0$ which had breakdown at the interval $(t, t + dt)$. Analogously $J_-(\sigma, t)$ is a ratio of $|R(t)/v(0)|$ for the trajectories which emerged after breakdown at $t = 0$. Finally, $M(t)dt$ gives us the average ratio of the velocities $v(t)/v(0)$ between the two breakdowns which happened at time 0 and in the interval $(t, t + dt)$. Note that the normalization factor σ^{-3} in the definition of $J_+(\sigma_0, t)$ accounts for the flow of trajectories given by $(\sigma^2 + \sigma\tau^{-1})Z$. With such a normalization one has $\int_\Omega dt J_+(\sigma_0, t)$ is the average ratio of $|v(t_k)/R(0)|$ of all trajectories which satisfy $\sigma(0) = \sigma_0$ and had a single breakdown at time $t_k \in \Omega$. After introducing these three new functions we are able to

average the ratio $|R(t)/R(0)|$ over all trajectories with an arbitrary number of breakdowns happened at all possible times. We can write formally

$$\left\langle \frac{|R(t)|}{|R(0)|} \right\rangle = \int d\sigma \left[Z(\sigma,\sigma_0,t) + \sum_{N=1}^{\infty} \int \prod_{k=1}^{N} dt_k J_+(\sigma_0,t_1) J_-(\sigma,t_k) \prod_{j=1}^{N-1} M(t_{j+1}-t_j) \right] \quad (22)$$

This expression can be simplified by turning to the Laplace transform representation:

$$\left\langle \frac{|R(t)|}{|R(0)|} \right\rangle = \int \frac{d\sigma ds}{2\pi i} \exp(st) \left\{ Z^s + J_+^s J_-^s + J_+^s M^s J_+^s + \dots \right\} = \int \frac{d\sigma ds}{2\pi i} \exp(st) \left\{ Z^s + \frac{J_+^s J_-^s}{1-M^s} \right\} \quad (23)$$

Where the upper index s corresponds to the Laplace transform of a function:

$$F^s = \int_0^\infty dt \exp(-st) F(t) \quad (24)$$

Long time asymptotic of both expressions is determined by the most left pole or other singularity of the integrated functions. One can easily note all three functions Z^s, M^s and $J_\pm^s$ have the same poles $s = E_k$. Therefore the long time asymptotic is determined either by E_0 or the most left solution of the equation $M^s = 1$. We will show that in our cases the asymptotic is indeed determined by the later singularities. First we want to show that $M^0 = -1$. The Laplace transform of Z obeys the following equation:

$$\left[s - \partial_\sigma \left(\frac{\sigma}{\tau} + \sigma^2 \right) - D\partial_\sigma^2 - \sigma \right] Z_\sigma = \delta(\sigma - \sigma_0) \quad (25)$$

here we have inserted the initial conditions $Z(\sigma, t=0) = \delta(\sigma - \sigma_0)$ in the r.h.s. explicitly. Remarkably this equation may be rewritten in the divergent form after the substition $Z = (\tau^{-1}+\sigma)^{-1}\Pi$. This fact was probably first noted in [20]. New equation acquires the following form:

$$\left[s + \partial_\sigma \left\{ -\sigma(\tau^{-1}+\sigma) + \frac{D}{\tau^{-1}+\sigma} \right\} - D\partial_\sigma^2 \right] \Pi = (1+\sigma_0)\delta(\sigma-\sigma_0) \quad (26)$$

In order to find Π^0 we have to set $s = 0$ in this equation, after which it can be easily integrated:

$$Z^0(\sigma,\sigma_0) = \frac{1}{D} \frac{\tau^{-1}+\sigma_0}{\tau^{-1}+\sigma} U(\sigma) \int_{-\infty}^{\sigma'} U^{-1}(\sigma_1) d\sigma_1 \quad (27)$$

where $\sigma' = \min(\sigma,\sigma_0)$ and $U(\sigma) = (\tau^{-1}+\sigma)^2 \exp(-\sigma^2/2D\tau - \sigma^3/3D)$. Straightforward integration of (27) yields the expected result:

$$M^0 = \lim \left(-\frac{\sigma^3}{\sigma_0} \right) Z^0(\sigma,\sigma_0) = -1 \quad (28)$$

where we assumed the limit $\sigma \to -\infty, \sigma_0 \to +\infty$. Returning now to the determination of long time asymptotic of $|R(\sigma, t)|$ we conclude that it is given by the solution of the equation $M^s = 1$. In general case the explicit form of M^s can not be found analytically, so in the next consideration we will assume the limit $D\tau^3 \ll 1$. In this case we know that the growth rate of $|R|$ is parametrically small, so that the solution s is almost zero. So, we need to know what is the behaviour of M^s near zero. In order to analyze it we turn the equation (26). After the substitution $\Pi = U^{1/2}(\sigma)\Psi$ we arrive to the quantum-mechanical problem

$$\hat{H}\Psi = -s\Psi, \qquad \hat{H} = -D\partial_\sigma^2 + \frac{\sigma^2(\tau^{-1}+\sigma)^2}{4D} - \frac{1}{2\tau} - 2\sigma \tag{29}$$

Such asymmetric double-well Hamiltonians have been extensively studied in the literature, see e.g. [16, 20] where the spectrum of $\hat{H}$ was analyzed in the limit $D \to 0$. Omitting the details we will just note that in the main order the ground state energy of $\hat{H}$ is negative, with absolute value $E_0 = -E = -(D\tau^2/2\pi)\exp(-1/6D\tau^3)$ while the other energy levels are positive and are of order unity (are not exponentially small). From the hermiticity of $\hat{H}$ it follows that the general form of M^s will be the following:

$$M^s = \sum_k \frac{c_k}{s + E_k} \tag{30}$$

where $c_k = -\sigma^2 U^{1/2}(\sigma)U^{-1/2}(\sigma_0)\Psi_k(\sigma)\Psi_k(\sigma_0)$ taken at limits $\sigma \to -\infty, \sigma_0 \to +\infty$. All c_k in the main order are proportional to $c_k \propto \exp(-1/6D\tau^3)$ and are thus exponentially small, while out of all energies E_k only E_0 is exponentially small. Therefore in the vicinity of $s = 0$ only the term with $k = 0$ is relevant. Although we could find the c_0 *ab initio* we won't do that and will instead use the fact that $M^0 = -1$, which immediately yields $c_0 = E$. Therefore in order to find the expression for the growth rate of $|R|$ we have to solve the algebraic equation $E/(s - E) = 1$ from which we finally obtain the growth rate exponent of $|R|$.

$$\tilde{\gamma}_1\tau = s\tau = 2E\tau = (St/\pi)\,\exp(-1/6St)\ . \tag{31}$$

This final expression shows that indeed the leading singularity in (23) is determined by the solution of $M^s = 1$. The growth rate $\tilde{\gamma}_1$ is exponentially small because it is determined by the rare breakdown events. Let us emphasize that we have established asymptotically exact pre-exponential factor in (31).

We now present the results of numerical simulations of the growth of particle separation $< |R|^k >$ in Lagrangian frame and of negative moments of density $< n^{-k} >$ in Eulerian

frame. The method used to obtain the growth rates is the Multicanonical Monte Carlo [23], a technique of adaptive importance sampling which boosts the probability of rare events that determine large negative moments. The Lagrangian results were obtained solving (1). The results presented in Figs.2,3 confirm an exponential growth of $\langle |R|^k \rangle$.

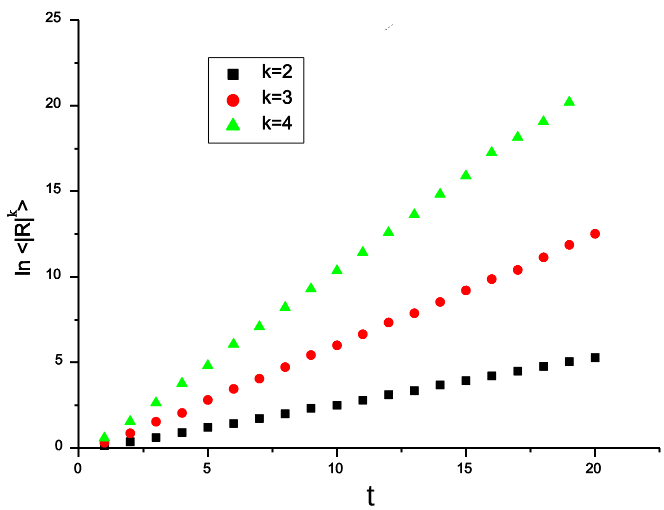

FIG. 2: The moments $< |R|^k >$ for $k = 2, 3, 4$ for $St = 0.2$. Time is normalized by τ.

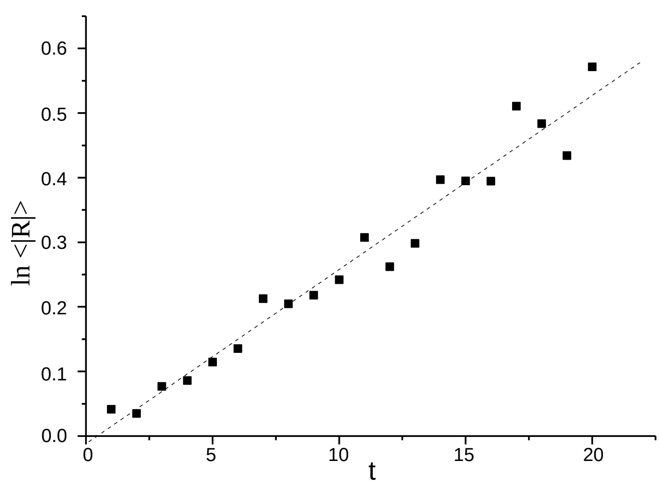

FIG. 3: Because of inertia the modulus of particle separation $< |R| >$ grows ($St = 0.2$).

We also observe an exponential growth of the particle separation, $< |R| >$. Figure 4 shows a good agreement between the numerics and the theoretical prediction (31) up to a fairly large $St \simeq 0.35$.

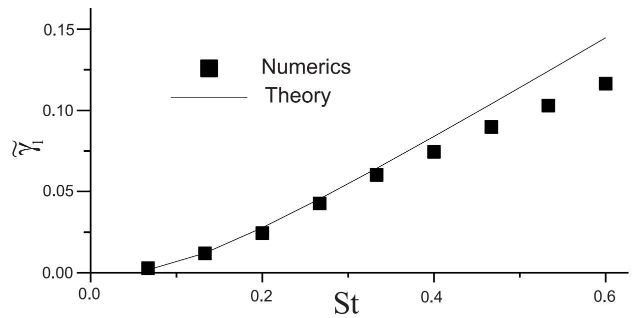

FIG. 4: The growth rate $\tilde{\gamma}_1$ vs the Stokes number. The solid curve represents theoretical prediction.

For the 1D Eulerian simulations the density field is given by the following expression

$$n(x,t) = \int dx_0 n_0(x_0)\delta(x(t|x_0) - x) \tag{32}$$

where $n_0(x_0)$ is an initial Eulerian density distribution (which we assume uniform) and $x(t|x_0)$ is a Lagrangian trajectory of a particle.

This trajectory is obtained from the system of the ODEs (characteristic equations):

$$\frac{d}{dt}x(t|x_0) = v(t), \quad x(0|x_0) = x_0, \tag{33}$$

$$\frac{d}{dt}v(t) = -\frac{v(t) - u(x(t|x_0),t)}{\tau} \tag{34}$$

where $u(x,t)$ is the Eulerian Gaussian velocity of the turbulent flow.

We assume that it is delta-correlated in time and has a spatial correlation length l_c:

$$< u(x,t)u(x',t') >= B(x-x')\delta(t-t'), \quad B(x) = B_0 \mathrm{e}^{-x^2/l_c^2} \tag{35}$$

The specific form of the correlation function B is not important. Eulerian field $u(x,t)$ is related to the Lagrangian process $s(t)$ (see Eq. (2)) via $s(t) = \partial u(x(t|x_0),t)/\partial x$. From this it follows that $St \equiv D\tau^3 = (\tau/2)\,|B''(0)| = (\tau/l_c^2)\,B_0$. Prior to solving system of ODEs (33), (34) one has to generate 1D Eulerian velocity field $u(x,t)$ with the prescribed correlation function (35). The algorithm for this is fairly standard (See. e.g. [24]). First we notice that since the field $u(x,t)$ is delta correlated in time its temporal regularization is trivial. Introducing discrete temporal step Δt at each time step, n, we now need to generate spatially distributed Gaussain field $u_n(x)$ with the correlation property $< u_n(x)u_m(x') >= B(x-x')\,\delta_{mn}$. In order to generate the field $u_n(x)$ we utilise the Fourier method. Indeed

the field $u_n(x)$ can be represented as a following Fourier integral

$$u_n(x) = \int_0^\infty \cos(2\pi kx)\,[2E(k)]^{1/2}\xi_n(k)dk + \int_0^\infty \sin(2\pi kx)\,[2E(k)]^{1/2}\eta_n(k)dk \tag{36}$$

where $\xi_n(k)$ and $\eta_n(k)$ are independent real Gaussian processes with the following properties:

$$\begin{gathered}\langle \xi_n(k) \rangle = \langle \eta_n(k) \rangle = 0 \\ < \xi_n(k)\xi_m(k') > = \langle \eta_n(k)\eta_m(k') \rangle = \delta(k-k')\,\delta_{mn}\end{gathered} \tag{37}$$

and $E(k)$ is an energy spectrum of the random field u_n, it coincides with the Fourier transform of the correlation function $B(x)$:

$$E(k) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}kx} B(x)dx = B_0\,\sqrt{\pi l_c^2}\,\exp\left[-\pi^2 l_c^2 k^2\right] \tag{38}$$

We then use a discrete version of (36):

$$u_n(x) \approx \sqrt{E(0)\,\Delta k}\,\xi_0^n + \sum_{j=1}^{M} \sqrt{2\,E(k_j)\,\Delta k}\,\left[\xi_j^n \cos(2\pi k_j\,x) + \eta_j^n \sin(2\pi k_j\,x)\right] \tag{39}$$

Here we have partitioned the Fourier space into M intervals, so that the wavevectors $k_j = j\Delta k$ denote the locations of the equispaced grid points. Variables ξ_j^n and η_j^n form a set of independent standard Gaussian variables (mean zero and unit variance). Because of the nature of the Fourier method the synthetically generated field $u_n(x)$ will contain an intrinsic spatial period $\lambda_F = (\Delta k)^{-1}$. Naturally one wishes to make it much bigger than the characteristic scale of the system L. On the other hand one has to ensure that all we have enough harmonics in (39) to sample the peak of the function $E(k)$. These two requirements can be met assuming $(l_c M)^{-1} \lesssim \Delta k \ll L^{-1}$.

Once we generated the synthetic Eulerian velocity field $u(x,t)$ we use a method of Lagrangian markers to obtain the Eulerian particle density at each point (using effectively formula (32)). We introduce a chain of N_L representative Lagrangian markers connected by some fictitious "strings". Each "string" contains a large constant number of uniformly distributed real particles. This number is fixed for each string, it does not change during the evolution and is determined by the initial density distribution. During the evolution, the strings deform according to the Lagrangian dynamics of the initial markers. In particular

the occurrence of explosions in Lagrangian frame corresponds to the formation of *folds* in the chain of markers. In order to obtain numerically the local Eulerian particle density at a given point we count the number of strings passing through this point and then for each string determine the contribution to the density as a ratio N_i/l_i where N_i is the number of particles in the string and l_i is the current length of the string. In Fig. 5 we plot the first four negative moments of n. Similarly to Lagrangian moments, Eulerian moments also grow exponentially: $< n^{-k+1} > \propto \exp(\Gamma_k t)$. The table compares Γ_k and Lagrangian $\tilde{\gamma}_k$ given by (17) for $St = 0.1$ and $St = 0.2$. We see that Lagrangian breakdowns (Eulerian folds) violate $\Gamma_k = \tilde{\gamma}_k$ that one would have for a smooth flow. We do not have a meaningful parametrization for the dependencies of $\tilde{\gamma}_k - \Gamma_k$ on k and St. It is likely that rare explosions cannot be completely disentangled from the exponential evolution.

k	$\tilde{\gamma}_k$	Γ_k	$\tilde{\gamma}_k - \Gamma_k$	k	$\tilde{\gamma}_k$	Γ_k	$\tilde{\gamma}_k - \Gamma_k$
1	0.006	—	—	1	0.028	—	—
2	0.158	0.146	0.012 ± 0.003	2	0.274	0.250	0.025 ± 0.002
3	0.393	0.374	0.019 ± 0.005	3	0.643	0.611	0.032 ± 0.005
4	0.695	0.666	0.029 ± 0.006	4	1.098	1.054	0.044 ± 0.008
5	1.054	1.012	0.043 ± 0.009	5	1.627	1.564	0.063 ± 0.009
6	1.459	1.403	0.056 ± 0.010	6	2.223	2.131	0.098 ± 0.012

TABLE I: The comparison of Eulerian and Lagrangian growth rates for $St = 0.1$ (left) and $St = 0.2$ (right).

In 1D case there is a very simple way of visualizing the dynamics of the caustics. At

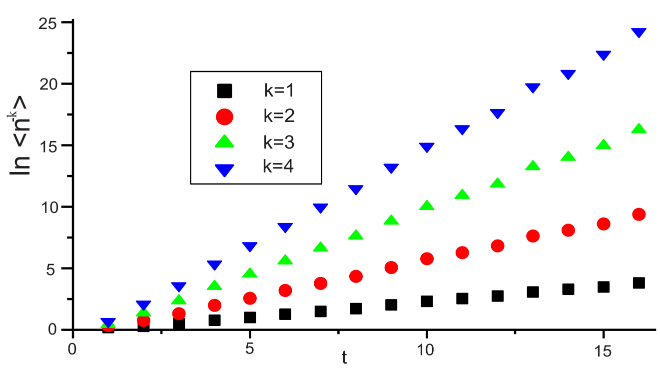

FIG. 5: The Eulerian moments $< n^{-k} >$ for $St = 0.2$.

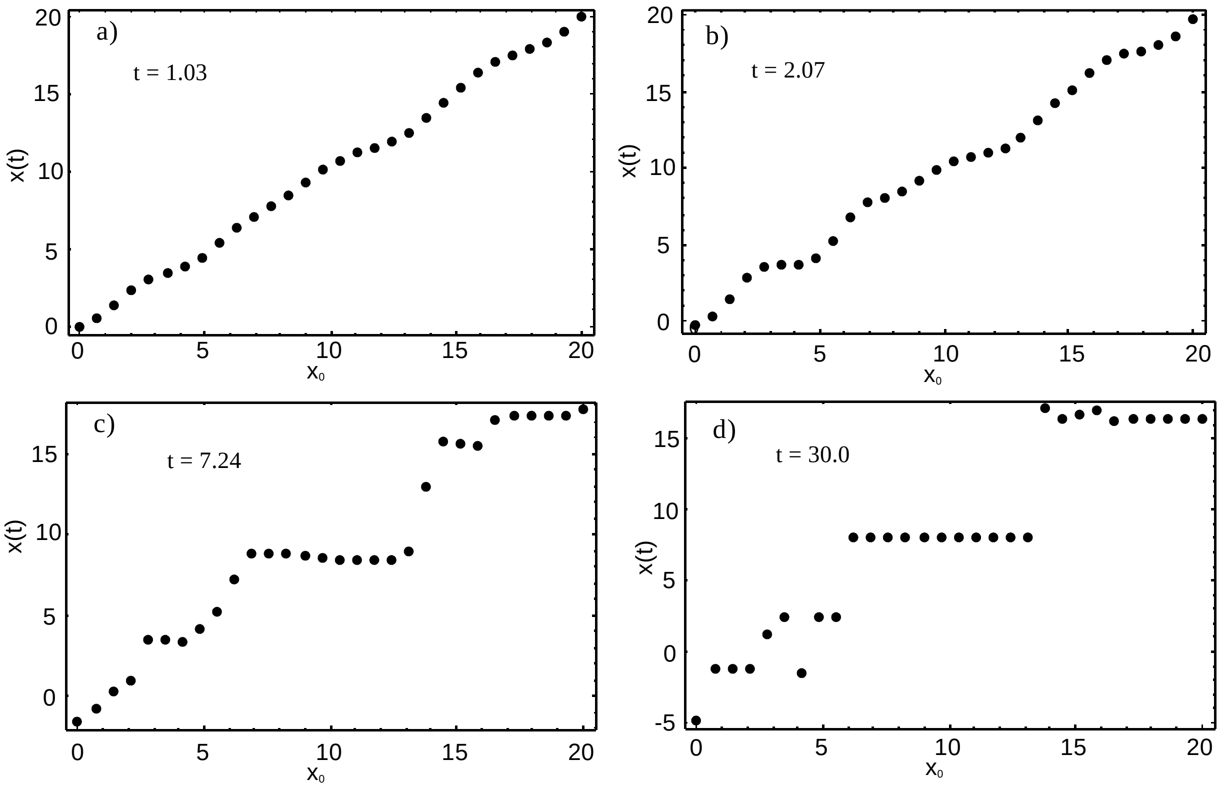

FIG. 6: The evolution of 30 Lagrangian markers with time. The time is normalized to τ and increases from a) to d). $St = 0.4$

a given time moment, t, one can plot the final displacement of a particle, $x(t)$ versus its initial position x_0. The Eulerian density distribution can be obtained by projecting the plot onto the coordinate axis $x(t)$. At the time moment $t = 0$ the curve is just a straight line at the half the right angle to the axes. During the evolution it will deform, according to the Lagrangian dynamics of individual particles (33),(34) eventually leading to the formation of folds illustrating the nonlocal nature of Eulerian density.

In Fig.6 we plot three stages of evolution of particle distribution. We take $N_L = 30$ initially equispaced Lagrangian markers and follow the evolution of the function $x(x_0)$ through time for a particular realization of the velocity field. We observe that at the initial stage (Fig.6a) the particle displacements are small so that the density distribution is smooth and there is one-to-ne correspondence $x(t) \leftrightarrow x_0$. Fig.6b shows the appearance of the first caustic (a particle overtakes another). At Fig.6c the folds are more pronounced and clearly visible. Finally at large times (Fig.6.d) one can evidently observe the effect of the clustering of particles.

Let us summarize the peculiarities of the evolution of the distribution of inertial particles that distinguish them from smooth compressible flows: 1) Infinite moments of density and inter-particle distance may appear non-analytically at $t = +0$; 2) Average distance between particles grows exponentially; 3) Moments of density in the Eulerian reference frame grow with the rates not reducible to those of distance moments in the Lagrangian frame. The work was supported by the Israeli Science Foundation, the EPSRC and the Royal Society.

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